Closed-Loop Control Performance Sensitivity to Parameter Variations

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A very efficient technique for computing the closed-loop performance sensitivities to parameter variations of a dynamic system with a reduced-order controller has been developed. The eigensystem of the closed-loop system is computed once. With this information, the closed-loop filter and state rms responses, and the first and second derivatives of these rms values with respect to given parameters are computed. Detailed numerical examples using the flexible beam at the Jet Propulsion Laboratory and a 55-m offset-fed, wrap-rib antenna are included.

I. Introduction

THE possibility of orbiting large space structures (LSS) in the relatively near future has provoked intense research into the control of these spacecraft. The "new" control problems that have emerged from the study of LSS may be broadly characterized as 1) those due to the high-order dynamic models, and 2) those due to uncertainties in the dynamic models. In the first category it is very likely that onboard controllers will be based upon greatly simplified models of the actual dynamic systems. This constraint is imposed mainly by the computational ability of the onboard computer. As a result, some control system performance is sacrificed for simple implementation of the control system. In the second category, a large class of LSS will not support their own weight in the 1g environment. Consequently, accurate dynamic models of the fully deployed spacecraft will not be available. This deficiency will also result in initially degraded control system performance.

Closed-loop control performance sensitivity is a classical subject, and there have been numerous publications in the general area. Jacobi[1] and Aubrun[2] developed techniques for evaluating eigenvalue and eigenvector sensitivities to parameter changes. Kriendler[3] and Sesak[4] have sought to reduce the closed-loop control sensitivity to parameter variation by careful controller design. At the other end of the spectrum, Mishne[5] has exploited highly sensitive controllers for their ability to identify system parameters. The purpose of the following work is to develop a direct and computationally efficient tool for predicting closed-loop control performance sensitivity (as measured by the statistical steady-state rms response) to parameter variations. This approach allows for the explicit computation of derivatives of the rms value of particular state vector components with respect to given parameters. The approach used here is made quite general by assuming the system to be represented in state variable format.

$$\dot{x}=Fx+Gu+\Gamma w \qquad z=Hx+v$$
$$\dot{x}_T=F_Tx_T+G_Tu+K(z-H_Tx_T) \qquad u=-Cx_T \tag{1}$$

where x is the state vector, u the control vector, z the measurement vector, w the state disturbance, with zero mean and spectral density Q, v the measurement noise, with zero mean and spectral density R, and x_T the truncated filter state.

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There are fairly standard approaches, OPTSYS (Ref. 6), for obtaining optimal control gains C, and optimal estimator gains K, based upon minimization of quadratic performance indices. The approaches are generally based upon full state feedback. Quite often, as is the case with the control of large structures, reduced-order controllers must be employed to minimize computational time and storage onboard a spacecraft. Whether or not the controller is of reduced order, it is also sometimes desirable to compute the closed-loop performance sensitivity to particular parameters. Although this can be accomplished by exhaustively studying a rather large set of perturbed problems, a much more efficient approach can be employed to calculate *directly* the closed-loop control performance sensitivity to parameter variation. This will be discussed in the next section.

II. Direct Computation of Closed-Loop Performance Sensitivity to Parameter Variation

This section contains the analysis for computing directly and efficiently the closed-loop performance and the first and second derivatives of the closed-loop performance with respect to given parameters.

A. Problem Formulation

Let the expressions in Eq. (1) be written compactly as follows:

$$\begin{bmatrix}\dot{x}\\ \dot{x}_T\end{bmatrix}=\begin{bmatrix}F & -GC\\ KH & F_T-G_TC-KH_T\end{bmatrix}\begin{bmatrix}x\\ x_T\end{bmatrix}+\begin{bmatrix}\Gamma & 0\\ 0 & K\end{bmatrix}\begin{bmatrix}w\\ v\end{bmatrix} \tag{2}$$

Now, rewriting Eq. (2) so that

$$x=\begin{bmatrix}x\\ x_T\end{bmatrix} \qquad A=\begin{bmatrix}F & -GC\\ KH & F_T-G_TC-KH_T\end{bmatrix}$$
$$\Gamma=\begin{bmatrix}\Gamma & 0\\ 0 & K\end{bmatrix} \qquad w=\begin{bmatrix}w\\ v\end{bmatrix} \qquad Q=\begin{bmatrix}Q & 0\\ 0 & R\end{bmatrix} \tag{3}$$

we obtain

$$\dot{x}=Ax+\Gamma w \tag{4}$$

and the statistical steady-state response of Eq. (4) is given in terms of the Lyapunov equation:

$$AX+XA^T+\Gamma Q\Gamma^T=0 \tag{5}$$

if the matrix A is stable. The diagonal elements of the covariance matrix X give the performance measures of the closed-loop control system. Note that the solution of Eq. (5) yields the covariance of the filter and state variables using a reduced-order filter.

B. Computation of the Derivatives of the Closed-Loop Performance with Respect to Given Parameters

It is often desirable, especially in the case of reduced-order controllers, to compute the closed-loop performance sensitivity to parameter changes. Of course, this may be performed by slightly perturbing the original system, and solving the Lyapunov equation in (5) for a variety of perturbed systems. A much shorter process is available. Consider A, X, and Γ to be functions of some parameter, p. Then, formally differentiating Eq. (5) with respect to p yields,

$$A\frac{\partial X}{\partial p}+\frac{\partial X}{\partial p}A^T+\left(\frac{\partial A}{\partial p}X+X\frac{\partial A^T}{\partial p}+\frac{\partial \Gamma}{\partial p}Q\Gamma^T+\Gamma Q\frac{\partial \Gamma^T}{\partial p}\right)=0 \tag{6}$$

Differentiating a second time yields

$$A\frac{\partial^2 X}{\partial p^2}+\frac{\partial^2 X}{\partial p^2}A^T+\left(2\frac{\partial A}{\partial p}\frac{\partial X}{\partial p}+2\frac{\partial X}{\partial p}\frac{\partial A^T}{\partial p}+\frac{\partial^2 A}{\partial p^2}X\right.$$
$$\left.+X\frac{\partial^2 A^T}{\partial p^2}+\frac{\partial^2 \Gamma}{\partial p^2}Q\Gamma^T+2\frac{\partial \Gamma}{\partial p}Q\frac{\partial \Gamma^T}{\partial p}+\Gamma Q\frac{\partial^2 \Gamma^T}{\partial p^2}\right)=0 \tag{7}$$

Note that after solving Eq. (5) for the steady-state covariance, X, Eq. (6) is simply a Lyapunov equation for the unknown, $\partial X/\partial P$, which is the derivative of the steady-state covariance, X, with respect to the parameter p. Similarly, after having solved Eqs. (5) and (6) for X and $\partial X/\partial P$, Eq. (7) is simply a Lyapunov equation for the unknown $\partial^2 X/\partial p^2$.

The feature that makes Eqs. (6) and (7) easy to solve for their respective unknowns is that they are the same as the Lyapunov equation that was solved in Eq. (5), except with a new right-hand side. Since initially obtaining the eigensystem of A for the solution of Eq. (5) requires the bulk of the computational load, the subsequent solution for $\partial X/\partial p$ and $\partial^2 X/\partial p^2$ in Eqs. (6) and (7) is virtually free. Examples of the use of this technique are found in Sec. II.D.

C. Solution of the Lyapunov Equation

It was shown in Secs. II.A and II.B that the computation of the closed-loop performance and the sensitivities of closed-loop performance to parameter changes depends solely upon the solution of a steady-state Lyapunov equation. It is therefore worthwhile to explore the most efficient and stable techniques available for implementing the final algorithm. One such technique is described by Bartels.[7]

A more easily implemented algorithm is described for this work. Let T be the matrix of eigenvectors of A. Then premultiplying Eq. (5) by T^{-1} and postmultiplying Eq. (5) by T^{-T} yields

$$T^{-1}ATT^{-1}XT^{-T}+T^{-1}XT^{-T}T^TA^TT^{-T}+T^{-1}\Gamma Q\Gamma^TT^{-T}=0 \tag{8}$$

Letting

$$T^{-1}AT=\Lambda$$
$$T^{-1}XT^{-T}=Y$$
$$T^{-1}\Gamma Q\Gamma^TT^{-T}=Z \tag{9}$$

Eq. (6) becomes

$$\Lambda Y+Y\Lambda^T+Z=0 \tag{10}$$

where Λ is a block diagonal matrix with a maximal block order of 2, Z is known, and Y must be obtained. However, since Λ is block diagonal, the solution for Y is easily obtained. From this, the solution for X may be easily obtained from Eqs. (9). Before attempting to solve Eq. (10), the eigenvalues of A are checked to assure that the closed-loop system is stable. If the system is stable, the closed-loop filter and state covariance may be computed as described previously.

D. Examples

In this section two examples of the use of the previously described algorithm will be presented. The first example is based upon the control of the flexible beam at the Jet Propulsion Laboratory (JPL). A picture of this facility is given in Fig. 1. This experimental facility has been used extensively over the past four years for verifying many aspects of the control of LSS in a laboratory setting. A three-mode model was used for design and evaluation purposes.

The second example is the 55-m, offset-fed, wrap-rib antenna shown in Fig. 2. The dynamic model for this system consisted of three spacecraft modes, three boom hinge modes, and six dish modes. In this example, a reduced-order controller based upon a rigid body model was implemented, but the full 12-mode model was used for evaluation purposes.

In the control of the JPL flexible beam, we have the following system equations for the modal amplitudes of the first three flexible modes in the format given in Eqs. (1):

$$F=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -4.564 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -22.21 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -74.10 & 0 \end{bmatrix}$$

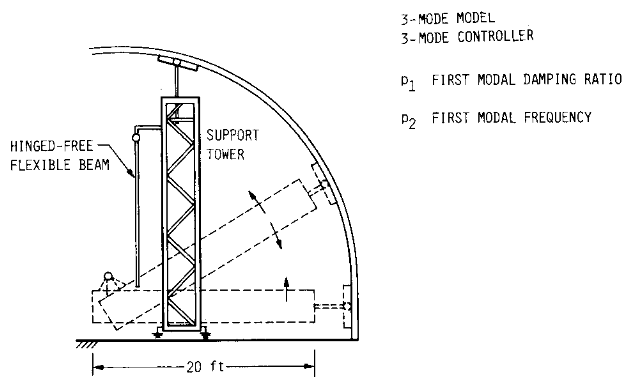

Fig. 1 Jet Propulsion Laboratory flexible beam control facility.

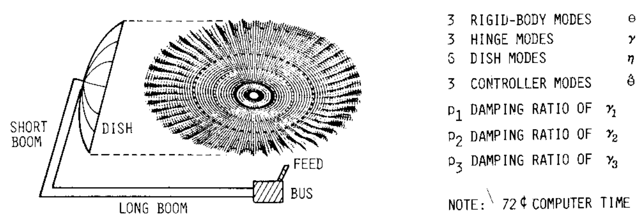

Fig. 2 55-m offset-fed antenna.

$$G=\Gamma=\begin{bmatrix} 0 \\ 0.3564 \\ 0 \\ 0.4432 \\ 0 \\ 0.4063 \end{bmatrix}$$

$$H=[3.782 \quad 0 \quad 4.704 \quad 0 \quad 4.311 \quad 0]$$

$$Q=2.0\times 10^{-4}, \qquad R=2.0\times 10^{-5}$$

The objective of the control system in this case is to damp the motion of the free end of the beam. Using OPTSYS, the following optimal control and estimator gains are obtained:

$$C=[-4.901, \ -7.576, \ -13.374, \ -3.443, \ -11.393, \ -1.808]$$

$$K=\begin{bmatrix} 0.4427 \\ 0.6131 \\ 0.2738 \\ 0.5476 \\ 0.1436 \\ 0.3517 \end{bmatrix}$$

Note that F consists of decoupled second-order blocks in the form:

$$\begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix}$$

Suppose it is desired to find the derivatives of the closed-loop, steady-state, state rms response with respect to the two parameters ξ_1 and ω_1. Then $\partial A/\partial p$ and $\partial \Gamma/\partial p$ consist of all zeros except in case 1,

$$\frac{\partial A(2,2)}{\partial \xi_1}=-2\omega_1=-4.2726$$

And in case 2,

$$\frac{\partial A(1,2)}{\partial \omega_1}=-2\omega_1=-4.2726, \qquad \frac{\partial A(2,2)}{\partial \omega_1}=-2\xi_1=0$$

A computer program has been developed for implementing the previously described sensitivity analysis. The program first gives the closed-loop eigenvalues and the steady-state rms values of the state variables and the filter variables. Next, the program gives the first and second derivatives of these values with respect to the parameters. These results are presented in Table 1. The total cost for executing the complete analysis of this 12-mode system with the first- and second-order sensitivity computations due to two parameters was $0.20.

In case 1 note that the first derivatives corresponding to mode 1 are negative, and all their corresponding second derivatives are positive. This simply says that increasing the damping in mode 1 will reduce the corresponding rms values of the state and filter. This result agrees with one's intuition. Furthermore, the minimum value for an rms value due to the increase in damping cannot continue forever. This fact is apparent since all of the corresponding second derivatives are positive.

The results in case 2 are interesting, although not very intuitive. If the frequency of mode 1 increases, some of the rms values of the state and filter increase and some decrease. This is precisely the type of information that is needed in sensitivity studies.

Although it is not possible to present performance sensitivity analysis results for perturbations in all the parameters of interest of the 55-m antenna, three parameters have been singled out as having special interest. These parameters are the damping ratios of the boom hinges. Rather than presenting the rather large matrices involved with modeling and controlling this antenna, the results of the analysis, and a discussion of these results are included below.

The evaluation model for the antenna consists of 12 modes—three spacecraft modes, three hinge modes, and six dish modes. All flexible modes are assumed to have 0.5% of critical damping open loop. The controller made use of only the rigid body modes, and the objective of the controller is to maintain the attitude of the antenna to 0.02 deg.

The results of this analysis are included in Table 2. The first eigenvalue on the list corresponds to the first dish mode, η_1. Although this mode was assumed to have 0.5% open-loop damping, spillover effects have pushed this mode to the limits of stability. As a result, the pointing of the spacecraft is well within the requirements, however, the surface of the dish may be somewhat degraded. As might be expected, the performance sensitivity to parameter variations can be expected to be quite large, as a result of the very lightly damped closed-loop mode. This is evident in Table 2 by the relatively large values associated with the performance of η_1 to all the parameter changes.

Table 1 Flexible beam results

	Re λ	Im λ	σ	$\frac{\partial\sigma}{\partial p_1}$	$\frac{\partial^2\sigma}{\partial p_1^2}$	$\frac{\partial\sigma}{\partial p_2}$	$\frac{\partial^2\sigma}{\partial p_2^2}$
x_1	−0.318152	8.552398	0.002381	−0.007037	−0.069540	−0.002622	0.010981
$\dot{x}_1$	−0.318152	−8.552398	0.004989	−0.014692	0.169925	0.003156	0.062318
x_2	−0.446394	8.617195	0.001488	−0.000577	0.007255	−0.000440	0.005406
$\dot{x}_2$	−0.446394	−8.617195	0.007158	−0.002099	0.037108	−0.001881	0.033265
x_3	−0.695434	2.647196	0.001081	0.000246	−0.001387	0.000249	−0.001466
$\dot{x}_3$	−0.695434	−2.647196	0.009350	0.001463	−0.012634	0.001429	−0.011376
$\hat{x}_1$	−0.776927	4.680054	0.001789	−0.003831	0.078107	−0.002420	0.042690
$\dot{\hat{x}}_1$	−0.776927	−4.680054	0.002863	−0.001738	0.067265	0.000132	0.005030
$\hat{x}_2$	−0.939318	4.774401	0.000988	0.003027	0.039560	0.003045	0.037657
$\dot{\hat{x}}_2$	−0.939318	−4.774401	0.004554	0.014280	0.147675	0.014038	0.143422
$\hat{x}_3$	−1.094543	2.468799	0.000691	−0.001714	0.015750	−0.001734	0.015793
$\dot{\hat{x}}_3$	−1.094543	−2.468799	0.005938	−0.015511	0.136272	−0.015726	0.136506

Table 2 55-m Antenna results

Parameters	Re λ	Im λ	σ	$\frac{\partial\sigma}{\partial p_3}$	$\frac{\partial\sigma}{\partial p_2}$	$\frac{\partial\sigma}{\partial p_3}$
θ_1	−0.000129	−0.391951	0.000078	0.049367	0.044790	0.109228
$\dot\theta_1$	−0.000129	−0.391951	0.000029	0.020536	0.018642	0.045312
θ_2	−0.002776	0.587595	0.000020	−0.000188	−0.000188	0.000188
$\dot\theta_2$	−0.002776	−0.587595	0.000001	−0.000030	−0.000030	−0.000030
θ_3	−0.005372	1.109705	0.000023	0.002035	0.001835	0.004644
$\dot\theta_3$	−0.005372	−1.109705	0.000004	0.002142	0.001945	0.004718
γ_1	−0.006294	1.229159	0.000082	0.054033	0.048463	0.117690
$\dot\gamma_1$	−0.006294	−1.229159	0.000034	0.020519	0.018134	0.043997
γ_2	−0.006614	1.267277	0.000000	−0.000000	0.000022	−0.000090
$\dot\gamma_2$	−0.006614	−1.267277	0.000000	−0.000000	0.000024	−0.000000
γ_3	−0.012101	1.568385	0.000283	0.200550	0.182034	0.442729
$\dot\gamma_3$	−0.012101	−1.568385	0.000111	0.078617	0.71360	0.173523
η_1	−0.014735	0.037133	0.011557	8.048633	7.305646	17.759764
$\dot\eta_1$	−0.014735	−0.037133	0.004999	2.860578	2.596396	6.309686
η_2	−0.022843	0.024850	0.001030	0.153756	0.142480	0.339291
$\dot\eta_2$	−0.022843	−0.024850	0.001131	0.021863	0.023846	0.047825
η_3	−0.023561	4.710167	0.003928	2.292814	2.080940	5.061717
$\dot\eta_3$	−0.023561	−4.710167	0.002587	0.536672	0.487077	1.182566
η_4	−0.026846	0.027543	0.000011	−0.000005	0.000498	−0.000005
$\dot\eta_4$	−0.026846	−0.027543	0.000013	−0.000001	0.000518	−0.000001
η_5	−0.044413	6.226265	0.000243	0.159894	0.143271	0.347896
$\dot\eta_5$	−0.044413	−6.226265	0.000101	0.059976	0.052892	0.128308
η_6	−0.110421	9.794254	0.000032	0.020709	0.018595	0.045059
$\dot\eta_6$	−0.110421	−9.794254	0.000016	0.006647	0.005963	0.014219
$\hat\theta_1$	−0.202407	0.028615	0.000039	0.015663	0.014285	0.033689
$\dot{\hat\theta}_1$	−0.202407	−0.028615	0.000003	0.001318	0.001207	0.002765
$\hat\theta_2$	−0.202700	0.023244	0.000012	−0.000296	−0.000296	−0.000296
$\dot{\hat\theta}_2$	−0.202700	−0.023244	0.000001	−0.000030	−0.000030	−0.000030
$\hat\theta_3$	−0.263797	0.130074	0.000013	0.000488	0.000467	0.000758
$\dot{\hat\theta}_3$	−0.263797	−0.130074	0.000001	0.000057	0.000054	0.000089

III. Conclusions

A very efficient program for obtaining the closed-loop eigenvalues, and, optionally, the steady-state filter and state rms values, and their first and second derivatives with respect to certain parameters, has been developed. The program is particularly valuable for determining closed-loop control performance as a function of parameter variations.

Acknowledgments

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